4.3: Counting with Functions

Question 1. A *quaternary string* is a string made up of 0's, 1's, 2's, and 3's. Devise a method for converting a quaternary string of length n to a binary string of length $2n$, and vice versa. (Start with the case of $n = 2$.) If possible, ensure that your conversion methods return the original string when composed.

One-to-One Correspondences.

Theorem 1. *Let $|X| = m$ and $|Y| = n$. If there is some function $f : X \to Y$ that is one-to-one, then $m \leq n$.*

Theorem 2. *Let $|X| = m$ and $|Y| = n$. If there is some function $f : X \to Y$ that is onto, then $m \geq n$.*

Corollary 3. *Let $|X| = m$ and $|Y| = n$. If there is a one-to-one correspondence $f : X \to Y$, then $m = n$.*

Example 1. In a single-elimination tournament, players are paired up in each round, and the winner of each match advances to the next round. If the number of players in a round is odd, one player gets a bye to the next round. The tournament continues until only two players are left; these two players play the championship game to determine the winner of the tournament. In a tournament of 270 players, how many games must be played?

Example 2. Six lines are drawn such that every line intersects every other line and no three lines intersect in a single point. How many triangles are formed?

Definition 4. *A function $f : X \to Y$ is called $n - to - one$ if every y in the image of the function has exactly n different elements of X that map to it. In other words, f is n-to-one if*

$$|\{x \in X | f(x) = y\}| = n \quad \textit{for all } y \in Y.$$

Example 3. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{0, 1\}$. Define a function $m : X \to Y$ by

$$m(x) = x \mod 2.$$

This function is 3-to-one.

Theorem 5. *Let $|X| = p$ and $|Y| = q$. If there is an n-to-one function $f : X \to Y$ that maps X onto Y, then $p = qn$.*

Example 4. Prove that $P(n, r) = r! \cdot C(n, r)$.

Example 5. How many different strings can you form by rearranging the letters in the word ENUMERATE?

Example 6. A group of 10 people sit in a circle around a campfire. How many different seating arrangements are there? Let us agree that a seating arrangement is determined only by the neighbors of each person, not by where on the ground they sit or the orientation of the circle.

The Pigeonhole Principle.

Theorem 6. *Let* $|X| = n$ *and* $|C| = r$, *and let* $f : X \to C$. *If* $n > r$, *then there are distinct elements* $x, y \in X$ *with* $f(x) = f(y)$.

Example 7. In a club with 400 members, must there be some pair of members who share the same birthday?

Example 8. Chandra has a drawer with 12 red and 14 green socks. He must grab a selection of clothes in the dark to avoid waking his roommate. How many socks must he grab to be assured of having a matching pair?

Example 9. In a round-robin tournament, every player plays every other player exactly once. Prove that, if no player goes undefeated, at the end of the tournament there must be two players with the same number of wins.

The Generalized Pigeonhole Principle.

Theorem 7. *Let $|X| = n$ and $|C| = r$, and let $f : X \to C$. If $n > r(l-1)$, then there is some subset $U \subseteq X$ such that $|U| = l$ and $f(x) = f(y)$ for all $x, y \in U$.*

Corollary 8. *Let $|X| = n$ and $|C| = r$, and let $f : X \to C$. Then there is some subset $U \subseteq X$ such that*

$$|U| = \rceil \frac{n}{r} \lceil$$

and $f(x) = f(y)$ for any $x, y \in U$.

Example 10. A website displays an image each day from a bank of 30 images. In any given 100-day period, show that some image must be displayed four times.

Example 11. Let G be a complete graph on six vertices. Suppose the 15 edges of this graph are colored red or green. Show that there must be some triangular circuit whose edges are the same color.

No Homework.
Practice Problems. Section 4.3: 3-13, 18-23, 28, 30